The Water Jug Puzzle

There are three water jugs, which can hold 8, 5 and 3 gallons of water respectively. The first jug is full, and the other two are empty. It is necessary to divide the water into two equal portions, that is, to put exactly four gallons into both the 8 and 5 gallon jugs. The only operation allowed is to pour some or all of the contents of one jug into another. Can the water be divided into two equal portions? How? Can the water be divided into portions of 4, 2 and 2 gallons? How? The answer is on the next two pages.

Solution to "The Water Jug Puzzle"

Yes, the water can be divided into two equal portions:

(8, 0, 0)	0: Beginning
(3, 5, 0)	1: 5 gallons from #1 into #2.
(3, 2, 3)	2: 3 gallons from #2 into #3.
(6, 2, 0)	3: 3 gallons from #3 into #1.
(6, 0, 2)	4: 2 gallons from #2 into #3.
(1, 5, 2)	5: 5 gallons from #1 into #2.
(1, 4, 3)	6: 1 gallon from #2 into #3.
(4, 4, 0)	7: 3 gallons from #3 into #1. DONE!

This cannot be done in fewer steps.

No, it is impossible to divide the water into portions of 4, 2 and 2 gallons. To see why, consider the following diagram of the 24 possible arrangements of the water in the jugs:

800					
701	710				
602	611	620			
503	512	521	530		
...	413	422	431	440	
...	...	323	332	341	350
...	...	...	233	242	251
...	...	...	...	143	152
...	...	...	...	...	053

The rules for pouring actually define a directed graph or "digraph" that points from one arrangement to the others that can be reached from it. Starting from the arrangement 800, we can find all new arrangements reachable in one pouring. Now we can look at all new arrangements reachable from those arrangements, and so on. This is a "breadth first" search of the digraph. The results are:

800			
350	503		1 Pouring
053	323	530	2 Pourings
620	233		3 Pourings
602	251		4 Pourings
152	701		5 Pourings
143	710		6 Pourings
440	413		7 Pourings

(no more arrangements are reachable)

Thus, there are 16 "reachable" arrangements out of the 24.

Solution to "The Water Jug Puzzle" - Continued

From this list, we see that:

It takes seven pourings to get from 800 to 440, and there can be no shorter solution.
There is no way to divide the water into portions of 4, 2, and 2 gallons, since the arrangements 422 and 242 are not reachable from 800.

Using this methodical and exhaustive method of looking at the problem, it should be easy to set up and solve similar problems, determining whether there is a way of reaching the desired arrangement, the shortest number of steps, and the actual sequence of steps.

On the other hand, you should have been able to say, almost immediately, that the 422 solution is impossible. Why? How did the third jug end up with just 2 gallons? If it's full or empty, that's easy, but if it's partially full, that can only happen because:

The entire contents of another jug were poured in, or
The contents of jug 3 were poured into another jug until it was filled.
Neither one of these cases can have occurred, because no jug is full or empty. And in fact, this is why the set of reachable solutions is the "hollowed out" part of the previous diagram:

800					
701	710				
602	xxx	620			
503	xxx	xxx	530		
...	413	xxx	xxx	440	
...	...	323	xxx	xxx	350
...	...	...	233	xxx	251
...	...	...	...	143	152
...	...	...	...	...	053

That is, the arrangements on the boundary represent cases where we have completely filled or emptied at least one jug, and the forbidden interior is where each jug is neither full nor empty.

Reference:

http://orion.math.iastate.edu

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